Instabilities in two flavor quark matter

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Abstract. I discuss briefly the instabilities of two flavor quark matter, paying attention to the gradient instability which develops in the g2SC phase in the Goldstone $U(1)_A$ sector.

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It is widely accepted that at high density and low temperature, the ground state of deconfined quark matter is the color superconductor [1, 2, 3]. In this talk I consider two flavor superconductive deconfined quark matter, which consists of u and d massless quarks, whose action is given by

$$S=\int d^4x\left[\bar{\psi}_{i\alpha}\left(i\gamma^{\mu}\partial_{\mu}\delta^{\alpha\beta}\delta_{ij}+\mu_{ij}^{\alpha\beta}\gamma_0\right)\psi_{j\beta}+(L\to R)+\mathscr{L}_{\Delta}\right] . \tag{1}$$

In the above equation, ψ denotes left-handed fields; Greek (Latin) indices stem for color (flavor); $\mu_{ij}^{\alpha\beta}$ is the chemical potential matrix, depending on the mean quark chemical potential μ, the charge chemical potential $\mu_Q=-\mu_e$ and the color chemical potentials μ_3, μ_8. Condensation in the quark-quark channel is described by the lagrangian $\mathscr{L}_{\Delta}$ which is given in the mean field approximation by

$$\mathscr{L}_{\Delta}=-\frac{\Delta}{2}\psi_{i\alpha}^{T}C\psi_{j\beta}\varepsilon^{\alpha\beta 3}\varepsilon_{ij}+H.c.-(L\to R) , \tag{2}$$

with C the charge conjugation matrix. We assume that in the ground state

$$\langle\psi_{i\alpha}^{L}C\psi_{j\beta}^{L}\rangle=-\langle\psi_{i\alpha}^{R}C\psi_{j\beta}^{R}\rangle\propto\Delta\varepsilon^{\alpha\beta 3}\varepsilon_{ij}\neq 0 , \tag{3}$$

where the superscripts L,R denote left-handed and right-handed quarks respectively. Eq. (3) means that pairs u_r-d_g, u_g-d_r are formed, with zero total momentum and zero total spin; on the other hand the blue quarks do not have a role in the pairing phenomenon.

The quark chemical potential matrix is given by

$$\boldsymbol{\mu}=(\mu\mathbf{1}_F-\boldsymbol{Q}\mu_e)\otimes\mathbf{1}_C+\mathbf{1}_F\otimes(\mu_3\boldsymbol{T}_3+\mu_8\boldsymbol{T}_8) . \tag{4}$$

In the ground state described by Eq. (3) one has $\mu_3=0$ and $\mu_8=\mathscr{O}(\Delta^2/\mu)$; on the other hand, $\mu_e=\mathscr{O}(0.1\mu)\gg\mu_8$. Therefore in what follows we assume $\mu_8=0$ in order to simplify the calculations. We stress that even considering $\mu_8\neq 0$ does not change the results presented here, since the eight color chemical potential does not change the

difference of the chemical potentials between the paired quarks, which are the relevant ones in this context. With this choice one has $\mu_u = \bar{\mu} - \delta\mu$ and $\mu_d = \bar{\mu} + \delta\mu$, with $\delta\mu = \mu_e/2$. Hence, once $\bar{\mu}$ is fixed, only $\delta\mu$ is needed to specify the spectrum of the system. The dispersion laws of the paired quarks are

$$E_{\pm\pm} = \left| \pm\delta\mu \pm \sqrt{(p-\bar{\mu})^2 + \Delta^2} \right| , \tag{5}$$

where $\bar{\mu}$ is the mean chemical potential and Δ is the gap parameter. It is easily realized that if $\delta\mu > \Delta$ then the dispersion law (5) has two nodes, and is thus gapless. When $\delta\mu < \Delta$ the phase is the 2SC; on the other hand if $\delta\mu > \Delta$ the phase is the gapless 2SC (g2SC) phase, considered in the QCD context in [4].

Neglecting electromagnetism, the symmetry group of two massless flavor QCD at high chemical potential (axial symmetry is unbroken at high density),

$$G_{QCD} = SU(3)_c \otimes U(2)_V \otimes U(2)_A ,$$

is broken down by the quark condensate $\langle\psi\psi\rangle$ to

$$G_{2SC} = SU(2)_c \otimes U(2)_V \otimes SU(2)_A .$$

There is a broken $U(1)_A$ since the diquark is not invariant for a phase shift of the quark fields. Then, from the Goldstone theorem it follows that one massless scalar appears in the spectrum, corresponding to the breaking of $U(1)_A$. The color group is also broken: as a consequence five of the eigth gluons become massive (Meissner effect, familiar from ordinary superconductivity). In this talk I focus on the Goldstone mode related to $U(1)_A$, discussing briefly the Meissner effect.

The Goldstone field ϕ describes small fluctuations of the condensate around its mean field value. It can be introduced in the model by the replacement, in the quark lagrangian, $\langle\psi\psi\rangle \to e^{2i\phi/f}\langle\psi\psi\rangle$. Integration over the quark fields in the functional integral gives rise to the one loop effective action of ϕ; it can be written in the low energy regime $p \ll \Delta$ as [5]

$$\mathscr{L}(p) = \frac{1}{2}\left[p_0^2\phi^2 - v^2(\boldsymbol{p}\phi)\cdot(\boldsymbol{p}\phi)\right] . \tag{6}$$

Evaluation of the loop integrals gives

$$f^2 = \frac{4\mu^2}{\pi^2}\left(1 - \theta(\delta\mu - \Delta)\frac{\sqrt{\delta\mu^2 - \Delta^2}}{\delta\mu}\right) ,$$

obtained by the requirement of canonical normalization of the lagrangian, and

$$v^2 = \frac{1}{3}\theta(\Delta - \delta\mu) - \frac{1}{3}\theta(\delta\mu - \Delta)\frac{\delta\mu}{\sqrt{\delta\mu^2 - \Delta^2}} . \tag{7}$$

Eq. (7) shows that $v^2 < 0$ in the g2SC phase. We thus have a *gradient instability* of ϕ.

The condensate in Eq. (3) breaks $SU(3)_c$ down to $SU(2)_c$: as a consequence, five of the eight gluons are massive (Higgs mechanism). In particular, one can evaluate the

Meissner masses, defined by $m_M^2 = -\Pi(p_0 = 0, \boldsymbol{p} \to 0)$ where Π is the polarization tensor of the gluons. One finds [6]

$$m_{M,4}^2 = \frac{4\alpha_s\mu^2}{3\pi}\left(\frac{\Delta^2 - 2\delta\mu^2}{2\Delta^2} + \frac{\delta\mu\sqrt{\delta\mu^2-\Delta^2}}{\Delta^2}\theta(\delta\mu - \Delta)\right) , \tag{8}$$

$$m_{M,8}^2 = \frac{4\alpha_s\mu^2}{9\pi}\left(1 - \frac{\delta\mu}{\sqrt{\delta\mu^2-\Delta^2}}\right) . \tag{9}$$

One notices the relation $f^2v^2 \propto m_{M,8}^2$, linking the instability in the Goldstone sector to the one in the gluon sector. Moreover we notice that in the interval $(\Delta/\sqrt{2}) \leq \delta\mu \leq \Delta$ an instability in the sector of the gluons $a = 4,\ldots,7$ occurs. However it cannot be related to the Goldstone velocity instability since in this interval $v^2 > 0$. This is an evidence of the different nature of the instabilities between the gluons with $a = 4,\ldots,7$ and the gluon with $a = 8$.

Beside Meissner instability, the dispersion laws of dynamical gluons have been studied [7]: it was found that the gluons with $a = 4,\ldots,7$ have a negative squared plasmon mass for $\delta\mu \leq \Delta/\sqrt{2}$ and a positive squared velocity. On the other hand, the gluon with $a = 8$ is massless and has a negative squared velocity. It is interesting to notice that these kind of instabilities are found both for the electric and for the magnetic gluon modes, while the Meissner instability develops only for the magnetic gluons.

The result $v^2 < 0$ in Eq. (7) can be interpreted as $\langle\boldsymbol{\nabla}\phi\rangle \neq 0$. From the ansatz

$$\phi(t,\boldsymbol{x}) = \boldsymbol{\Phi}\cdot\boldsymbol{x} + h(t,\boldsymbol{x}) , \tag{10}$$

and assuming $\langle\boldsymbol{\nabla}h\rangle \neq 0$ [1] we get $\langle\boldsymbol{\nabla}\phi\rangle = \boldsymbol{\Phi}$, and we call $\boldsymbol{\Phi}$ the Goldstone current. The bosonization of the quark lagrangian is done via the transformation $\langle\psi\psi\rangle \to e^{2i\boldsymbol{\Phi}\cdot\boldsymbol{x}/f}\langle\psi\psi\rangle e^{2ih/f}$. The value Φ_0 of $|\boldsymbol{\Phi}|$ in the ground state is evaluated by minimizing the thermodynamic potential Ω. Expanding the quark propagator in powers of Δ/q with $q \equiv |\boldsymbol{\Phi}|/f$ we get $\Phi_0^2 \approx 1.2 \times f^2\delta\mu^2$. Moreover, expanding in the small field h/f and evaluating the quark loops we find

$$\mathscr{L}[h] = \frac{1}{2}\left((\partial_0 h)^2 - v_i v_j \partial_i h \partial_j h\right) , \tag{11}$$

and the low energy parameters are $(\Phi_x = \Phi_y = 0, \Phi_z = \Phi_0)$

$$f^2 \approx 0.46\mu^2\Delta^2/\delta\mu^2 , \quad \boldsymbol{v} = (0,0,1) . \tag{12}$$

Therefore the tensor $v_i v_j$ is semidefinite positive, signaling the Goldstone stability of the new ground state. The breaking of the rotational symmetry due to $\boldsymbol{\Phi} \neq 0$, $SO(3) \to SO(2)$, is reflected by the anisotropy of the tensor $v_i v_j$.

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[1] The assumption $\langle\boldsymbol{\nabla}h\rangle \neq 0$ is justified *a posteriori* since we find that the squared velocity of the fluctuation field h is positive, see Eq. (12).

The phase shift in the quark lagrangian, in the case of $\mathbf{\Phi} \neq 0$, is equivalent to consider an inhomogeneous superconductive state with gap $\Delta e^{2i\mathbf{\Phi}\cdot\boldsymbol{x}/f}$: this is the one-plane-wave (1PW) LOFF state [8], where the fluctuation field h plays the role of the $U(1)_A$ Goldstone mode. From the previous results we know that the Goldstone mode in the one-plane-wave state does not suffer the velocity instability. Moreover, the Meissner masses of the gluons in the 1PW state are positive (at least for small $\Delta/\delta\mu$, see below). But now we are faced with a problem: the stability criterion is a necessary but not a sufficient condition for the existence of a given phase, since one needs to compute the free energy in order to establish if this state is the ground state or not. Since there is an equivalence between the state with the Goldstone current (with ansatz given by (10)) and the one-plane-wave LOFF state, the free energy of the two phases is the same. From the LOFF literature we know that the free energy of the 1PW state yields it to be the ground state of two flavor quark matter, in the weak coupling limit, if

$$0.707\Delta_0 \leq \delta\mu \leq 0.754\Delta_0 \ , \quad \Delta_0 \equiv \Delta(\delta\mu = 0) \ .$$

This is a very narrow range. As a consequence, although the 1PW state satisfies the stability conditions, its free energy does not allow for the existence of this state in a wide interval of $\delta\mu$. Hence the 1PW state is unlikely to be the ground state of QCD. At this point, since we need a new state that is stable and has a lower free energy, we can either look for different a ansatz of the Goldstone current, or improve the 1PW state by adding more plane waves. The latter case is easier to be treated, although in this case we lose the correspondence with the Goldstone current state. We follow the latter philosophy, leaving the search for a different current ansatz to a future project.

In the multiple-plane-wave (MPW) state the ansatz for the gap is given by [9, 10]

$$\langle \psi\psi \rangle \propto \Delta \sum_{a=1}^{P} e^{2i\boldsymbol{q}^a\cdot\boldsymbol{x}} \ . \tag{13}$$

The low energy effective action for the $U(1)_A$ Goldstone mode is obtained following the same procedure adopted in the case of the 1PW state. The result in configuration space is again given by Eq. (11), with the squared velocity tensor being defined at the leading order in Δ/q as [5] (see also [11] for a similar calculation)

$$v_i v_j = \sum_{a=1}^{P} (\hat{q}^a)_i (\hat{q}^a)_j / P \ ; \tag{14}$$

it is an easy task to prove that it is enough to consider a structure with three orthogonal wave vectors in order to yield a definite positive tensor $v_i v_j$. Hence the Goldstone stability requirement is fulfilled in the MPW state. For example we find for the 1PW state

$$v_i v_j = [\mathrm{diag}(0,0,1)]_{ij} \ , \qquad \text{1PW}, \tag{15}$$

while for the FCC structure (corresponding to $P = 8$ in Eq. (13), with the wave vectors pointing to the edges of a cube) we find

$$v_i v_j = \frac{1}{3}\delta_{ij} \ , \qquad \text{MPW}. \tag{16}$$

Moreover, the Meissner stability requirement is also satisfied in the MPW state. As a matter of fact, at the leading order in Δ/q we find [5, 12],

$$\left(\mathscr{M}_{44}^{ij}\right)^2 = \frac{f^2}{16}v_i v_j \ , \tag{17}$$

$$\left(\mathscr{M}_{88}^{ij}\right)^2 = \frac{f^2}{12}v_i v_j \ . \tag{18}$$

and $\mathscr{M}_{55}^{ij} = \mathscr{M}_{66}^{ij} = \mathscr{M}_{77}^{ij} = \mathscr{M}_{44}^{ij}$. It is evident that the positivity of the Meissner tensor $\left(\mathscr{M}_{ab}^{ij}\right)^2$ follows from the positivity of the tensor $v_i v_j$.

We discuss now the interval of $\delta\mu$ in which the LOFF state is stable respect to the normal phase. We have found that the BCC structure ($P = 6$, the wave vectors pointing to the faces of a cube) is stable in the interval $0.71\Delta_0 \leq \delta\mu \leq 0.95\Delta_0$; on the other hand the FCC structure is favored in the range $0.95\Delta_0 \leq \delta\mu \leq 1.35\Delta_0$. Therefore LOFF state in the MPW configuration is stable in the interval

$$0.71 \leq \frac{\delta\mu}{\Delta_0} \leq 1.35 \ . \tag{19}$$

For larger values of $\delta\mu/\Delta_0$ the LOFF state is no longer energetically favored, and cannot solve the instability problem of high density QCD.

In conclusion, I have shown that beside chromomagnetic instabilities, homogeneous gapless color superconductive quark matter suffers of a Goldstone instability too, related to the negative squared velocity of the Goldstone mode. My results suggest that inhomogeneous superconductive states can solve this problem, if $\delta\mu$ lies in the interval (19). Beside them, gluonic phases [13, 14, 15] may have a relevant role to solve the instability puzzle of high density QCD. It should be noticed that nowadays a comparison of the free energies of the LOFF state (in the MPW state) and of the gluonic phases is still lacking (for studies about the comparison of the free energies of the one plane wave and the gluon phases see [16]). My results suggest that such a comparison should be done as soon as possible, as from it we could learn much more we know about the ground state of neutral superconductive quark matter.

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